# 2017 AIME I Problems

## Problem 1

Fifteen distinct points are designated on : the 3 vertices , , and ; other points on side ; other points on side ; and other points on side . Find the number of triangles with positive area whose vertices are among these points.

Solution

Every triangle is uniquely determined by 3 points. There are ways to choose 3 points, but that counts the degenerate triangles formed by choosing three points on a line. There are invalid cases on segment , invalid cases on segment , and invalid cases on segment for a total of invalid cases. The answer is thus .

## Problem 2

When each of , , and is divided by the positive integer , the remainder is always the positive integer . When each of , , and is divided by the positive integer , the remainder is always the positive integer . Find .

Solution

Let's tackle the first part of the problem first. We can safely assume: Now, if we subtract two values: which also equals Similarly, Since is the only common factor, we can assume that , and through simple division, that .

Using the same method on the second half: Then. The common factor is , so and through division, .

The answer is

## Problem 3

For a positive integer , let be the units digit of . Find the remainder when is divided by .

Solution

We see that appears in cycles of , adding a total of each cycle. Since , we know that by , there have been cycles, or has been added. This can be discarded, as we're just looking for the last three digits. Adding up the first of the cycle of , we get that the answer is .

## Problem 4

A pyramid has a triangular base with side lengths , , and . The three edges of the pyramid from the three corners of the base to the fourth vertex of the pyramid all have length . The volume of the pyramid is , where and are positive integers, and is not divisible by the square of any prime. Find .

Solution

Let the triangular base be , with . We find that the altitude to side is , so the area of is .

Let the fourth vertex of the tetrahedron be , and let the midpoint of be . Since is equidistant from , , and , the line through perpendicular to the plane of will pass through the circumcenter of , which we will call . Note that is equidistant from each of , , and . Then,

Let . Equation : Squaring both sides, we have

Substituting with equation :

We now find that .

Let the distance . Using the Pythagorean Theorem on triangle , , or (all three are congruent by SSS):

Finally, by the formula for volume of a pyramid,

This simplifies to , so .

## Problem 5

A rational number written in base eight is , where all digits are nonzero. The same number in base twelve is . Find the base-ten number .

Solution 1

First, note that the first two digits will always be a positive number. We will start with base twelve because of its repetition. List all the positive numbers in base twelve that have equal twelves and ones digits in base 8.

We stop because we only can have two-digit numbers in base 8 and 101 is not a 2 digit number. Compare the ones places to check if they are equal. We find that they are equal if or . Evaluating the places to the right side of the decimal point gives us or . When the numbers are converted into base 8, we get and . Since , the first value is correct. Compiling the necessary digits leaves us a final answer of

Solution 2

The parts before and after the decimal points must be equal. Therefore and . Simplifying the first equation gives . Plugging this into the second equation gives . Multiplying both sides by 64 gives . and are both digits between 1 and 7 (they must be a single non-zero digit in base eight) so using , or . Testing these gives that doesn't work, and gives , and . Therefore

## Problem 6

A circle is circumscribed around an isosceles triangle whose two congruent angles have degree measure . Two points are chosen independently and uniformly at random on the circle, and a chord is drawn between them. The probability that the chord intersects the triangle is . Find the difference between the largest and smallest possible values of .

Solution

The probability that the chord doesn't intersect the triangle is . The only way this can happen is if the two points are chosen on the same arc between two of the triangle vertices. The probability that a point is chosen on one of the arcs opposite one of the base angles is , and the probability that a point is chosen on the arc between the two base angles is . Therefore, we can write This simplifies to Which factors as So . The difference between these is .

## Problem 7

For nonnegative integers and with , let . Let denote the sum of all , where and are nonnegative integers with . Find the remainder when is divided by .

Solution1

Let , and note that . The problem thus asks for the sum over all such that . Consider an array of 18 dots, with 3 columns of 6 dots each. The desired expression counts the total number of ways to select 6 dots by considering each column separately. However, this must be equal to . Therefore, the answer is .

-rocketscience

Solution 2(Major Bash)

Case 1: .

Subcase 1:

Subcase 2: $\binom{6}{1}\binom{6}{2}\binom{6}{3}=1800 \equiv 800 \pmod {1000}$$\binom{6}{1}\binom{6}{3}\binom{6}{4}=1800 \equiv 800 \pmod {1000}$$800+800+540+36=2176 \equiv 176 \pmod {1000}$ Subcase 3: $\binom{6}{2}\binom{6}{3}\binom{6}{5}=1800\equiv800\pmod{1000}$$800+225=1025\equiv25\pmod{1000}$

$923+176+25=1124\equiv124\pmod{1000}$ Case 2:

By just switching and in all of the above cases, we will get all of the cases such that is true. Therefore, this case is also $124\pmod{1000}$

Case 3: $\binom{6}{2}\binom{6}{2}\binom{6}{4}=3375\equiv375\pmod{1000}$$1+540+375+400=1316\equiv316\pmod{1000}$

Solution 3

Treating as , this problem asks for:

( \* ( \* ))

But, can be seen as the following combinatorial argument:

Choosing elements from a set of size is the same as splitting the set into two sets of size and choosing elements from one, from the other where <= <= .

Thus, such a procedure is simply .

Therefore, our answer is . As such, our answer is .

## Problem 8

Two real numbers and are chosen independently and uniformly at random from the interval . Let and be two points on the plane with . Let and be on the same side of line such that the degree measures of and are and respectively, and and are both right angles. The probability that is equal to , where and are relatively prime positive integers. Find .

Solution 1

Noting that and are right angles, we realize that we can draw a semicircle with diameter and points and on the semicircle. Since the radius of the semicircle is , if , then $\overarc{QR}$ must be less than or equal to .

This simplifies the problem greatly. Since the degree measure of an angle on a circle is simply half the degree measure of its subtended arc, the problem is simply asking:

Given such that , what is the probability that ? Through simple geometric probability, we get that .

The answer is

Solution 2(Trig Bash)

Put and with on the origin and the triangles on the quadrant. The coordinates of and is , . So = , which we want to be less then . So So we want , which is equivalent to or . The second inequality is impossible so we only consider what the first inequality does to our by box in the plane. This cuts off two isosceles right triangles from opposite corners with side lengths from the by box. Hence the probability is and the answer is

## Problem 9

Let , and for each integer let . Find the least such that is a multiple of .

Solution 1

Writing out the recursive statement for and summing them gives Which simplifies to Therefore, is divisible by 99 if and only if is divisible by 99, so needs to be divisible by 9 and 11. Assume that is a multiple of 11. Writing out a few terms, , we see that is the smallest that works in this case. Next, assume that is a multiple of 11. Writing out a few terms, , we see that is the smallest that works in this case. The smallest is .

Solution 2

$a\_n \equiv a\_{n-1} + n \pmod {99}$ By looking at the first few terms, we can see that $a\_n \equiv 10+11+12+ \dots + n \pmod {99}$ This implies $a\_n \equiv \frac{n(n+1)}{2} - \frac{10\*9}{2} \pmod {99}$ Since $a\_n \equiv 0 \pmod {99}$, we can rewrite the equivalence, and simplify $0 \equiv \frac{n(n+1)}{2} - \frac{10\*9}{2} \pmod {99}$$0 \equiv n(n+1) - 90 \pmod {99}$$0 \equiv 4n^2+4n+36 \pmod {99}$$0 \equiv (2n+1)^2+35 \pmod {99}$$64 \equiv (2n+1)^2 \pmod {99}$ The only squares that are congruent to $64 \pmod {99}$ are and , so $2n+1 \equiv -8, 8, 19, \text{or } {-19} \pmod {99}$$2n+1 \equiv -8 \pmod {99}$ yields as the smallest integer solution.

$2n+1 \equiv 8 \pmod {99}$ yields as the smallest integer solution.

$2n+1 \equiv -19 \pmod {99}$ yields as the smallest integer solution.

$2n+1 \equiv 19 \pmod {99}$ yields as the smallest integer solution. However, must be greater than .

The smallest positive integer solution greater than is .

Solution 3

$a\_n=a\_{n-1} + n \pmod{99}$. Using the steps of the previous solution we get up to $n^2+n \equiv 90 \pmod{99}$. This gives away the fact that $(n)(n+1) \equiv 0 \pmod{9} \implies n \equiv \{0, 8\} \pmod{9}$ so either or must be a multiple of 9.

Case 1 (): Say and after simplification $x(9x+1) = 10 \pmod{90} \forall x \in \mathbb{Z}$.

Case 2: (): Say and after simplification $(9a-1)(a) = 10 \pmod{90} \forall a \in \mathbb{Z}$.

As a result must be a divisor of and after doing some testing in both cases the smallest value that works is .

## Problem 10

Let and where Let be the unique complex number with the properties that is a real number and the imaginary part of is the greatest possible. Find the real part of .

Solution 1

(This solution's quality may be very poor. If one feels that the solution is inadequate, one may choose to improve it.)

Let us write be some imaginary number with form Similarly, we can write as some

The product must be real, so we have that is real. Of this, must be real, so the imaginary parts only arise from the second part of the product. Thus we have

is real. The imaginary part of this is which we recognize as This is only when is some multiple of In this problem, this implies and must form a cyclic quadrilateral, so the possibilities of lie on the circumcircle of and

To maximize the imaginary part of it must lie at the top of the circumcircle, which means the real part of is the same as the real part of the circumcenter. The center of the circumcircle can be found in various ways, (such as computing the intersection of the perpendicular bisectors of the sides) and when computed gives us that the real part of the circumcenter is so the real part of is and thus our answer is

Solution 2 Algebra Bash

First we calculate , which becomes .

Next, we define to be for some real numbers and . Then, can be written as Multiplying both the numerator and denominator by the conjugate of the denominator, we get:

.

In order for the product to be a real number, since both denominators are real numbers, we must have the numerator of be a multiple of the conjugate of , namely So, we have and for some real number .

Then, we get:

Expanding both sides and combining like terms, we get:

which can be rewritten as:

Now, common sense tells us that to maximize , we would need to maximize . Therefore, we must set to its lowest value, namely 0. Therefore, must be

You can also notice that the ab terms cancel out so all you need is the x-coordinate of the center and only expand the a parts of the equation.

## Problem 11

Consider arrangements of the numbers in a array. For each such arrangement, let , , and be the medians of the numbers in rows , , and respectively, and let be the median of . Let be the number of arrangements for which . Find the remainder when is divided by .

Solution 1

We know that if is a median, then will be the median of the medians.

WLOG, assume is in the upper left corner. One of the two other values in the top row needs to be below , and the other needs to be above . This can be done in ways. The other can be arranged in ways. Finally, accounting for when is in every other space, our answer is . But we only need the last digits, so is our answer.

Solution 2 (Complementary Counting with probability)

Notice that m can only equal 4, 5, or 6, and 4 and 6 are symmetric.

WLOG let

There is a chance that exactly one of 1, 2, 3 is in the same row.

There is a chance that the other two smaller numbers end up in the same row.

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## Problem 12

Call a set product-free if there do not exist (not necessarily distinct) such that . For example, the empty set and the set are product-free, whereas the sets and are not product-free. Find the number of product-free subsets of the set .

Solution

We shall solve this problem by doing casework on the lowest element of the subset. Note that the number cannot be in the subset because . Let be a product-free set. If the lowest element of is , we consider the set . We see that 5 of these subsets can be a subset of (, , , , and the empty set). Now consider the set . We see that 3 of these subsets can be a subset of (, , and the empty set). Note that cannot be an element of , because is. Now consider the set . All four of these subsets can be a subset of . So if the smallest element of is , there are possible such sets.

If the smallest element of is , the only restriction we have is that is not in . This leaves us such sets.

If the smallest element of is not or , then can be any subset of , including the empty set. This gives us such subsets.

So our answer is .

## Problem 13

For every , let be the least positive integer with the following property: For every , there is always a perfect cube in the range . Find the remainder when is divided by 1000.

Solution

Lemma 1: The ratio between and decreases as increases.

Lemma 2: If the range includes cubes, will always contain at least cubes for all in .

If , the range includes one cube. The range includes 2 cubes, which fulfills the Lemma. Since also included a cube, we can assume that for all . Two groups of 1000 are included in the sum modulo 1000. They do not count since for all of them, therefore Now that we know this we will find the smallest that causes to contain two cubes and work backwards (recursion) until there is no cube in .

For there are two cubes in for . There are no cubes in but there is one in . Therefore .

For there are two cubes in for . There are no cubes in but there is one in . Therefore .

For in there are two cubes in for . There are no cubes in but there is one in . Therefore , and the same for , , and for a sum of .

For all other there is one cube in , , , and there are two in . Therefore, since there are 10 values of in the sum, this part sums to .

When the partial sums are added, we get $\boxed{059}\hspace{2 mm}QED\hspace{2 mm} \blacksquare$

## Problem 14

Let and satisfy and . Find the remainder when is divided by .

Solution

The first condition implies

So .

Putting each side to the power of :

so . Specifically,

so we have that

We only wish to find $x\bmod 1000$. To do this, we note that $x\equiv 0\bmod 8$ and now, by the Chinese Remainder Theorem, wish only to find $x\bmod 125$. By Euler's Theorem:

$2^{\phi(125)} = 2^{100} \equiv 1\bmod 125$ so

$2^{192} \equiv \frac{1}{2^8} \equiv \frac{1}{256} \equiv \frac{1}{6} \bmod 125$ so we only need to find the inverse of $6 \bmod 125$. It is easy to realize that $6\cdot 21 = 126 \equiv 1\bmod 125$, so

$x\equiv 21\bmod 125, x\equiv 0\bmod 8.$ Using CRT, we get that $x\equiv \boxed{896}\bmod 1000$, finishing the solution.

## Problem 15

The area of the smallest equilateral triangle with one vertex on each of the sides of the right triangle with side lengths and as shown, is where and are positive integers, and are relatively prime, and is not divisible by the square of any prime. Find

Solution 1

Lemma. If satisfy , then the minimal value of is .

Proof. Recall that the distance between the point and the line is given by . In particular, the distance between the origin and any point on the line is at least .

Let the vertices of the right triangle be and let be two of the vertices of the equilateral triangle. Then, the third vertex of the equilateral triangle is . This point must lie on the hypotenuse , i.e. must satisfy which can be simplified to By the lemma, the minimal value of is so the minimal area of the equilateral triangle is and hence the answer is .

Solution 2 (No Coordinates)

Let be the triangle with side lengths and .

We will think about this problem backwards, by constructing a triangle as large as possible (We will call it , for convenience) which is similar to with vertices outside of a unit equilateral triangle , such that each vertex of the equilateral triangle lies on a side of . After we find the side lengths of , we will use ratios to trace back towards the original problem.

First of all, let , , and (These three angles are simply the angles of triangle ; out of these three angles, is the smallest angle, and is the largest angle). Then let us consider a point inside such that , , and . Construct the circumcircles and of triangles and respectively.

From here, we will prove the lemma that if we choose points , , and on circumcircles and respectively such that , , and are collinear and , , and are collinear, then , , and must be collinear. First of all, if we let , then (by the properties of cyclic quadrilaterals), (by adjacent angles), (by cyclic quadrilaterals), (adjacent angles), and (cyclic quadrilaterals). Since and are supplementary, , , and are collinear as desired. Hence, has an inscribed equilateral triangle .

In addition, now we know that all triangles (as described above) must be similar to triangle , as and , so we have developed similarity between the two triangles. Thus, is the triangle similar to which we were desiring. Our goal now is to maximize the length of , in order to maximize the area of , to achieve our original goal.

Note that, all triangles are similar to each other if , , and are collinear. This is because is constant, and is also a constant value. Then we have similarity between this set of triangles. To maximize , we can instead maximize , which is simply the diameter of . From there, we can determine that , and with similar logic, , , and are perpendicular to , , and respectively We have found our desired largest possible triangle .

All we have to do now is to calculate , and use ratios from similar triangles to determine the side length of the equilateral triangle inscribed within . First of all, we will prove that . By the properties of cyclic quadrilaterals, , which means that . Now we will show that . Note that, by cyclic quadrilaterals, and . Hence, (since ), proving the aforementioned claim. Then, since and , .

Now we calculate and , which are simply the diameters of circumcircles and , respectively. By the extended law of sines, and .

We can now solve for with the law of cosines:

Now we will apply this discovery towards our original triangle . Since the ratio between and the hypotenuse of is , the side length of the equilateral triangle inscribed within must be (as is simply as scaled version of , and thus their corresponding inscribed equilateral triangles must be scaled by the same factor). Then the area of the equilateral triangle inscribed within is , implying that the answer is .

-Solution by TheBoomBox77

Solution 3

Let be the right triangle with sides , , and and right angle at .

Let an equilateral triangle touch , , and at , , and respectively, having side lengths of .

Now, call as and as . Thus, and .

By Law of Sines on triangles and ,

and .

Summing,

.

Now substituting , , and and solving, .

We seek to minimize .

This is equivalent to minimizing .

Using the lemma from solution 1, we conclude that

Thus, and our final answer is